

Integration Bee Solutions 2025

Round 1

1.1 $\int_0^1 \sqrt{1+3x} dx$

Solution. Let $u = 1 + 3x$. Then $du = 3 dx$, $dx = \frac{du}{3}$. We get

$$\int_0^1 \sqrt{1+3x} dx = \int_1^4 u^{1/2} \frac{du}{3} = \frac{1}{3} \cdot \frac{2}{3} u^{3/2} \Big|_1^4 = \frac{2}{9} (4^{3/2} - 1^{3/2}) = \frac{2}{9} (8 - 1) = \frac{14}{9}.$$

1.2 $\int_1^2 \frac{x^2-1}{x-1} dx$

Solution.

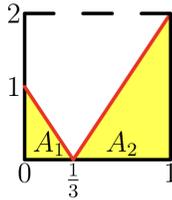
$$\int_1^2 \frac{x^2-1}{x-1} dx = \int_1^2 \frac{(x-1)(x+1)}{x-1} dx = \int_1^2 (x+1) dx = \left(\frac{x^2}{2} + x \right) \Big|_1^2 = (2+2) - \left(\frac{1}{2} + 1 \right) = \frac{5}{2}$$

1.3 $\int_0^1 |1-3x| dx$

Solution. The sign changes when $1 - 3x = 0$, which happens at $x = \frac{1}{3}$. We get

$$\int_0^1 |1-3x| dx = \int_0^{1/3} (1-3x) dx + \int_{1/3}^1 (3x-1) dx = \left(x - \frac{3}{2}x^2 \right) \Big|_0^{1/3} + \left(\frac{3}{2}x^2 - x \right) \Big|_{1/3}^1 = \frac{1}{6} + \frac{2}{3} = \frac{5}{6}$$

Alternatively, we can graph $y = 1 - 3x$ and observe that the integral is the sum of areas of the two triangles depicted in the diagram.



We get $\int_0^1 |1-3x| dx = A_1 + A_2 = \frac{1}{3} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{2}{2} = \frac{5}{6}$.

1.4 $\int_0^1 \frac{dx}{x^2+3x+2}$

Solution.

$$\begin{aligned} \int_0^1 \frac{dx}{x^2+3x+2} &= \int_0^1 \frac{dx}{(x+1)(x+2)} = \int_0^1 \left(\frac{1}{x+1} - \frac{1}{x+2} \right) dx = \ln \left(\frac{x+1}{x+2} \right) \Big|_0^1 \\ &= \ln \left(\frac{2/3}{1/2} \right) = \ln \frac{4}{3} \end{aligned}$$

1.5 $\int \frac{dx}{x^2+2x+2}$

Solution.

$$\int \frac{dx}{x^2+2x+2} = \int \frac{dx}{(x+1)^2+1} = \arctan(x+1) + C$$

1.6 $\int \frac{\tan x}{\cos^3 x} dx$

Solution. Let $u = \cos x$. Then $du = -\sin x dx$.

$$\int \frac{\tan x}{\cos^3 x} dx = \int \frac{\sin x}{\cos^4 x} dx = \int \frac{-du}{u^4} = \frac{1}{3u^3} + C = \frac{1}{3 \cos^3 x} + C$$

1.7 $\int_{-\pi}^{\pi} [2 + \sin(\sin x)] dx$

Solution.

$$\int_{-\pi}^{\pi} [2 + \sin(\sin x)] dx = \int_{-\pi}^{\pi} 2 dx + \int_{-\pi}^{\pi} \sin(\sin x) dx = 4\pi + 0 = 4\pi \quad \text{since } \sin(\sin x) \text{ is odd.}$$

1.8 **Problem.** $\int_0^3 \text{sign}(x^2 - 4) dx$

Solution. Since $x^2 - 4 < 0$ on $(0, 2)$ and $x^2 - 4 > 0$ on $(2, 3)$ we get

$$\int_0^3 \text{sign}(x^2 - 4) dx = \int_0^2 (-1) dx + \int_2^3 1 dx = -2 + 1 = -1$$

Round 2

2.1 **Problem.** $\int x^2 \ln x dx$

Solution. We integrate by parts. Let $u = \ln x$ and $dv = x^2 dx$. Then $du = \frac{dx}{x}$ and $v = \frac{x^3}{3}$. We get

$$\int x^2 \ln x dx = \frac{x^3}{3} \ln x - \int \frac{x^3}{3} \cdot \frac{1}{x} dx = \frac{x^3}{3} \ln x - \frac{1}{3} \int x^2 dx = \frac{x^3}{3} \ln x - \frac{x^3}{9} + C$$

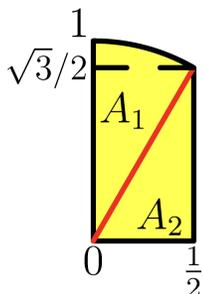
2.2 **Problem.** $\int \frac{x^3}{x^2+1} dx$

Solution.

$$\begin{aligned} \int \frac{x^3}{x^2+1} dx &= \int \left(\frac{x^3+x}{x^2+1} - \frac{x}{x^2+1} \right) dx = \int x dx - \int \frac{x}{x^2+1} dx = \frac{x^2}{2} - \frac{1}{2} \int \frac{d(x^2+1)}{x^2+1} \\ &= \frac{x^2}{2} - \frac{1}{2} \ln(x^2+1) + C \end{aligned}$$

2.3 **Problem.** $\int_0^{1/2} \sqrt{1-x^2} dx$

Solution. This integral represents the area under $y = \sqrt{1-x^2}$ from 0 to $1/2$.



Note that here A_1 is the area of the sector of the unit circle with angle $\arcsin(1/2) = \frac{\pi}{6}$. We get

$$\int_0^{1/2} \sqrt{1-x^2} dx = A_1 + A_2 = \frac{\pi}{12} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\pi}{12} + \frac{\sqrt{3}}{8}.$$

2.4 Problem. $\int \frac{\tan^3 x}{\cos^4 x} dx$

Solution.

$$\int \frac{\tan^3 x}{\cos^4 x} dx = \int \tan^3 x \sec^4 x dx = \int \tan^3 x (1 + \tan^2 x) \sec^2 x dx.$$

Let $u = \tan x$, $du = \sec^2 x dx$. Then

$$\int \tan^3 x (1 + \tan^2 x) \sec^2 x dx = \int (u^3 + u^5) du = \frac{u^4}{4} + \frac{u^6}{6} + C = \frac{\tan^4 x}{4} + \frac{\tan^6 x}{6} + C$$

Round 3

3.1 Problem. $\int_0^1 e^{\sqrt{x}} dx$

Solution.

Let $u = \sqrt{x}$, so $x = u^2$ and $dx = 2u du$ and u changes from 0 to 1. Then

$$\int_0^1 e^{\sqrt{x}} dx = 2 \int_0^1 u e^u du = 2(e^u(u-1)) \Big|_0^1 = 2(0 - (-1)) = 2.$$

3.2 Problem. $\int_0^1 \frac{1-2x \arctan x}{(1+x^2)^2} dx$

Solution.

We fit this to the quotient rule formula and notice that

$$\frac{d}{dx} \left(\frac{\arctan x}{1+x^2} \right) = \frac{(\arctan x)'(1+x^2) - (\arctan x)(1+x^2)'}{(1+x^2)^2} = \frac{1-2x \arctan x}{(1+x^2)^2}.$$

Hence

$$\int_0^1 \frac{1-2x \arctan x}{(1+x^2)^2} dx = \frac{\arctan x}{1+x^2} \Big|_0^1 = \frac{\pi/4}{2} - 0 = \frac{\pi}{8}.$$

3.3 Problem. $\int_0^{\pi/2} \sin^6 x + 3 \sin^4 x \cos^2 x dx$

Solution.

Using the change of variable $u = \frac{\pi}{2} - x$ we see that

$$\int_0^{\pi/2} \sin^6 x + 3 \sin^4 x \cos^2 x dx = \int_0^{\pi/2} 3 \sin^2 x \cos^4 x + \cos^6 x dx.$$

Hence

$$\begin{aligned}\int_0^{\pi/2} \sin^6 x + 3 \sin^4 x \cos^2 x \, dx &= \frac{1}{2} \int_0^{\pi/2} \sin^6 x + 3 \sin^4 x \cos^2 x + 3 \sin^2 x \cos^4 x + \cos^6 x \, dx \\ &= \frac{1}{2} \int_0^{\pi/2} (\sin^2 x + \cos^2 x)^3 \, dx = \frac{1}{2} \int_0^{\pi/2} 1 \, dx \\ &= \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}\end{aligned}$$